

A non-unitary trace formula for arbitrary groups

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Abstract: A trace formula for non-unitary twists is given for compact quotients of arbitrary locally compact groups. The geometric side of the trace formula is the same as in the unitary case, but on the spectral side one has to consider continuous Hilbert representations modulo continuous equivalence and the multiplicities are multiplicities as sub-quotients of $L^2(\Gamma \backslash G)$.

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Introduction

Werner Müller established in [Mül11] a trace formula with non-unitary twists for compact locally symmetric spaces $\Gamma \backslash G/K$. In the paper [DM16a] this was generalized to compact quotients $\Gamma \backslash G$ of a semi-simple Lie group G and in [DM16b] the spectral-theoretic groundwork is laid for the non-compact case.

In the present paper we extend [DM16a] to compact quotients $\Gamma \backslash G$, for an arbitrary locally compact group G . The geometric side of the trace formula stays as it is, where on the spectral side one has to consider continuous Hilbert representations modulo continuous equivalence and the multiplicities are multiplicities as sub-quotients of $L^2(\Gamma \backslash G)$.

In detail: given a locally compact group G , a cocompact lattice $\Gamma \subset G$ and a finite dimensional representation $\omega : G \rightarrow \mathrm{GL}(V)$, let $E = E_\omega$ denote the homogeneous vector bundle over $\Gamma \backslash G$ defined by ω . Assume first that ω is unitary. The unitary structure then yields a G -invariant hermitian metric on E and the G -action on the Hilbert space $L^2(E)$ defines a unitary representation $R = R_{\Gamma, \omega}$ of G which decomposes as a direct sum of irreducible representations

$$L^2(E) = \bigoplus_{\pi \in \widehat{G}} N_{\Gamma, \omega}(\pi),$$

where \widehat{G} is the unitary dual and $N_{\Gamma, \omega}(\pi)$ is the finite multiplicity of π as a sub-representation of $(R, L^2(E))$. Any $f \in C_c^\infty(G)$ then acts via integration and the induced operator $R(f)$ is trace class. Its trace can be computed by integration over the diagonal of its kernel, the comparison of the two calculations yields the *trace formula*

$$\sum_{\pi \in \widehat{G}} N_{\Gamma, \omega}(\pi) \operatorname{tr} \pi(f) = \sum_{[\gamma]} \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \mathcal{O}_\gamma(f) \operatorname{tr} \omega(\gamma),$$

where the sum on the right runs over all conjugacy classes $[\gamma]$ in the group Γ , the groups Γ_γ, G_γ are the centralizers of γ in Γ and G respectively. The orbital integral $\mathcal{O}_\gamma(f)$ is given by

$$\mathcal{O}_\gamma(f) = \int_{G_\gamma \backslash G} f(x^{-1} \gamma x) dx.$$

For details, see [DE09].

Now assume that the “twist” ω is no longer unitary. Still one can build the vector bundle E , but now there is no G -invariant metric on E . Choose an arbitrary non-invariant hermitian metric on E and consider the Hilbert space $L^2(E)$ of L^2 -sections. Then G acts on E and thus on the Hilbert space $L^2(E)$, the latter action defines a continuous representation R of G , which in general is not unitary. Let \tilde{G} denote the space of all irreducible Banach-representations modulo continuous equivalence. For each $\pi \in \tilde{G}$ let $N_{\Gamma, \omega}(\pi)$ denote the maximum of all $k \in \mathbb{N}_0$ such that there exist closed, G -stable subspaces

$$F'_1 \subset F_1 \subset F'_2 \subset F_2 \subset \cdots \subset F'_k \subset F_k$$

with the property that $F_j/F'_j \cong \pi$ for each j . It is the main result of this paper that for general ω and $f \in C_c^\infty(G)$ the operator $R_{\Gamma, \omega}(f)$ is trace class and that the trace formula holds with the multiplicities $N_{\Gamma, \omega}(\pi)$ and \hat{G} replaced with \tilde{G} .

1 Admissible representations

Suppose that G is a Lie group. Choose a left-invariant metric on G and let Δ denote the Laplace operator for this metric. We call such a Δ a *group-Laplacian*. Let $\mathfrak{g}_{\mathbb{R}}$ be the real Lie algebra of G and \mathfrak{g} its complexification. The universal enveloping algebra $U(\mathfrak{g})$ can be identified with the algebra of left-invariant differential operators on G , so Δ can be viewed as an element of $U(\mathfrak{g})$.

By a *representation* of G we mean a group homomorphism $R : G \rightarrow \mathrm{GL}(V)$ to the group of invertible bicontinuous linear operators on some Banach space V such that the map $G \times V \rightarrow V$, $(g, v) \mapsto \pi(g)v$ is continuous. The space of *smooth vectors* V^∞ then is defined as the space of all $v \in V$ such that $G \rightarrow V$, $x \mapsto R(x)v$ is infinitely differentiable. The universal enveloping algebra $U(\mathfrak{g})$ acts on the dense subspace V^∞ of smooth vectors.

Definition 1.1. A representation (R, V) of G is called Δ -*admissible*, if

- (a) there is a dense subset $\Lambda_R \subset \mathbb{C}$ with $\mathbb{R} \cap \Lambda_R = \emptyset$, such that for each $\lambda \in \Lambda_R$ the operator $R(\Delta - \lambda)^{-1}$ is defined and extends to a continuous operator on the space V . For every G -stable closed subspace $U \subset V$ one has $(\Delta - 1)^{-1}U \subset U$,

(b) for each $\sigma \in \mathbb{C}$ the generalized eigenspace

$$V(\Delta, \sigma) = \bigcup_{n \in \mathbb{N}} \ker(\Delta - \sigma)^n \subset V^\infty$$

is finite-dimensional,

(c) the set $\text{Spec}_R(\Delta)$ of all $\sigma \in \mathbb{C}$ with $V(\Delta, \sigma) \neq 0$ has no accumulation point in \mathbb{C} ,

(d) every $v \in V$ can be written as absolutely convergent sum

$$v = \sum_{\sigma \in \text{Spec}(\Delta)} v_\sigma,$$

each $v_\sigma \in V(\Delta, \sigma)$ is uniquely determined and the projection map $v \mapsto v_\sigma$ is continuous,

(e) for every $\sigma_0 \in \text{Spec}(\Delta)$ the space

$$V(\Delta, \sigma_0)' = \overline{\bigoplus_{\sigma \neq \sigma_0} V(\Delta, \sigma)}$$

satisfies $V = V(\Delta, \sigma_0) \oplus V(\Delta, \sigma_0)'$ and the operator $\Delta - \sigma_0$ has a bounded inverse on $V(\Delta, \sigma_0)'$.

The condition (a) needs explaining: We request that there exists a continuous operator T on V which preserves V^∞ as well as every G -stable closed subset and satisfies

$$TR(\Delta - \lambda)v = R(\Delta - \lambda)Tv = v$$

for every $v \in V^\infty$. We denote this operator by $R(\Delta, \lambda)^{-1}$.

We find it convenient to leave out the R in the notation, so we occasionally write $(\Delta - \lambda)$ instead of $R(\Delta - \lambda)$ and the same for the inverses.

Lemma 1.2. *Let (R, V) be Δ -admissible and let $U \subset V$ be a closed G -stable subspace, then U is Δ -admissible.*

Proof. The only part of the definition which needs proving, is part (d). More precisely we need to show that if $u \in U$ and $u = \sum_\sigma u_\sigma$ is the spectral

decomposition in V , then $u_\sigma \in U$ for every σ . For this let $\sigma_0 \in \text{Spec}_R(\Delta)$ and let $\lambda \in \Lambda_R$ be closer to σ_0 than any other $\sigma \in \text{Spec}_R(\Delta)$. Then the operator

$$T = (\sigma_0 - \lambda)(\Delta - \lambda)^{-1}$$

has eigenvalue 1 on $V(\Delta, \sigma_0)$ and eigenvalue of absolute value < 1 on $V(\Delta, \sigma)$ for every $\sigma_0 \neq \sigma \in \text{Spec}_R(\Delta)$. We write $u = u_{\sigma_0} + u^{\sigma_0}$, where $u^{\sigma_0} = \sum_{\sigma \neq \sigma_0} u_\sigma$. We first show that $T^n u^{\sigma_0}$ tends to 0 as $n \rightarrow \infty$. For this note that on the space $V(\Delta, \sigma_0)'$ one has

$$\begin{aligned} (\Delta - \lambda)^{-1} &= (\Delta - \lambda)^{-1} - (\Delta - \sigma_0)^{-1} + (\Delta - \sigma_0)^{-1} \\ &= (\sigma_0 - \lambda)(\Delta - \lambda)^{-1}(\Delta - \sigma_0)^{-1} + (\Delta - \sigma_0)^{-1}. \end{aligned}$$

Taking operator norms on both sides and using the triangle inequality we infer that for small values of $|\sigma_0 - \lambda|$ we have

$$\|(\Delta - \lambda)^{-1}\| \leq \frac{\|(\Delta - \sigma_0)^{-1}\|}{1 - (\sigma_0 - \lambda) \|(\Delta - \sigma_0)^{-1}\|},$$

where we mean the operator norm on the space $V(\Delta, \sigma_0)'$. It follows that for λ close enough to σ_0 the operator norm of T on $V(\Delta, \sigma_0)'$ is < 1 , which implies that $T^n u^{\sigma_0}$ tends to zero.

On $V(\Delta, \sigma_0)$ we write $\Delta = \sigma_0 - S$ where S is nilpotent. So

$$T = \left(1 + \sum_{j=1}^{N-1} \frac{S^j}{(\sigma_0 - \lambda)^j} \right) = (1 + R)$$

where $R = \sum_{j=1}^{N-1} \frac{S^j}{(\sigma_0 - \lambda)^j}$ is again nilpotent and $N = \dim V(\Delta, \sigma_0)$. Then on $V(\Delta, \sigma_0)$ we have

$$T^n = (1 + R)^n = \sum_{k=0}^{N-1} \binom{n}{k} R^k.$$

it follows that $T^n \binom{n}{N-1}^{-1}$ tends to R^{N-1} as $n \rightarrow \infty$, which implies that $R^{N-1} u_{\sigma_0}$ lies in U . Next $(T^n - \binom{n}{N-1}) R^{N-1} \binom{n}{N-2}^{-1}$ tends to R^{N-2} which implies that $R^{N-2} u_{\sigma_0}$ lies in U . We repeat until we reach $R^0 u_{\sigma_0} = u_{\sigma_0} \in U$ as claimed. \square

Definition 1.3. Let (R, V) be a representation of the locally compact group G and let π be an irreducible representation of G . A π -filtration in V is a sequence

$$F'_1 \subset F_1 \subset F'_2 \subset F_2 \subset \cdots \subset F'_l \subset F_l$$

of closed, G -stable subspaces such that $F_j/F'_j \cong \pi$ for each j .

Definition 1.4. We write \tilde{G} for the set of isomorphism classes of irreducible representations of G . Here we say that two continuous representations (π, V_π) and (η, V_η) are *isomorphic*, if there exists an invertible continuous linear operator $T : V_\pi \rightarrow V_\eta$ such that $T\pi(g) = \eta(g)T$ holds for every $g \in G$. Note that T^{-1} is also continuous by the open mapping theorem. Note also that if π and π' are irreducible, continuous and equivalent, then they are unitarily equivalent.

Definition 1.5. Let L be a linearly ordered set. If $i, j \in L$ such that $i < j$ and there is no $k \in L$ with $i < k < j$, then we say that i is a *left neighbor* of j and j is a *right neighbor* of i . A linearly ordered set L is called a *tower*, if every $j \in L$ has at least one neighbor.

Definition 1.6. A subset $I \subset L$ of a linearly ordered set L is called an *interval*, if

$$x < y < z, \quad x, z \in I \quad \Rightarrow \quad y \in I.$$

An interval I is called *connected*, if for any $x < z$ in I the set $\{y : x < y < z\}$ is finite. For $s \in L$ let $C(s)$ denote the union of all connected intervals containing s , then $C(s)$ is the largest connected interval containing s , called the *connected component* of s . A tower decomposes into its connected components. Any connected component is order isomorphic to $\pm\mathbb{N}, \mathbb{Z}$, or a finite interval.

Definition 1.7. Let L be a tower. Let (R, V) be a representation of a topological group G on a locally convex space V . A *complete* L -filtration on (R, V) is a family of closed, G -stable subspaces $(F_i)_{i \in L}$ such that

- (a) $F_i \subset F_j$ if $i \leq j$,
- (b) F_j/F_i is irreducible if $i < j$ are neighbored,
- (c) $\bigcup_{j \in L} F_j$ is dense in V and $\bigcap_{j \in L} F_j = 0$,

- (d) if $j \in L$ is not maximal and has no right neighbor, then $F_j = \bigcap_{k>j} F_k$,
- (e) if $j \in L$ is not minimal and has no left neighbor, then F_j is the closure of $\bigcup_{i<j} F_i$.

If V is the representation space of a representation of a topological group G , and the filtration is G -stable, then the filtration is said to have *irreducible quotients*, if for all neighbored pairs $i < j$ in L the quotient F_j/F_i is irreducible.

Theorem 1.8 (Spectral theorem). *Let (R, V) be a Δ -admissible representation of the Lie group G .*

- (a) *If $V_0 \subset V_1$ are closed G -stable subspaces, then the sub-quotient $S = V_1/V_0$ is Δ -admissible as well. Each spectral value λ of S is a spectral value of V , more precisely, one has*

$$S(\Delta, \lambda) \cong V_1(\Delta, \lambda)/V_0(\Delta, \lambda).$$

If $m(S, \lambda) = m(V, \lambda)$ for all λ , then $S = V$.

- (b) *Let (R, V) be Δ -admissible and π an irreducible representation of G . Then all maximal π -filtrations have the same finite length. We call this length $N_{\Gamma, \omega}(\pi) \in \mathbb{N}_0$ the multiplicity of π in R .*
- (c) *If $f \in C_c(G)$ is such that the operator $R(f) = \int_G f(x)R(x)dx$ is trace class, then $\pi(f)$ is trace class for every $\pi \in \tilde{G}$ with $N_{\Gamma, \omega}(\pi) > 0$ and one has*

$$\mathrm{tr} R(f) = \sum_{\pi \in \tilde{G}} N_{\Gamma, \omega}(\pi) \mathrm{tr} \pi(f).$$

- (d) *There exists a complete filtration on (R, V) .*

Proof. (a) A submodule is admissible, so it remains to show that a quotient is admissible. So let $U \subset V$ be a closed G -stable subspace. We claim that for $\lambda_0 \in \mathbb{C}$ the map $V(\Delta, \lambda_0) \rightarrow V/U$ induces an isomorphism $V(\Delta, \lambda_0)/U(\Delta, \lambda_0) \cong (V/U)(\Delta, \lambda_0)$. The injectivity is clear. For the surjectivity let $v + U$ be in $(V/U)(\Delta, \lambda_0)$, then $(\Delta - \lambda_0)^n v \in U$ for some n . Write $v = \sum_{\lambda \in \mathbb{C}} v_\lambda$ as in the definition of admissibility. We claim that $v - v_{\lambda_0}$ lies

in U . Let $\xi \in \mathbb{C} \setminus \text{Spec}_R(\Delta)$. Write $(\Delta - \lambda_0)^n v = \sum_{\lambda} w_{\lambda} \in U$, then each w_{λ} lies in U and

$$(\Delta - \xi)^{-n} \underbrace{(\Delta - \lambda_0)^n v}_{\in U} = (\Delta - \xi)^{-n} \sum_{\lambda} w_{\lambda} = \sum_{\lambda} (\Delta - \xi)^{-n} w_{\lambda} \in U.$$

which implies $(\Delta - \xi)^{-n} w_{\lambda} \in U$ the uniqueness of the λ -expansion.

For $\lambda \neq \lambda_0$ we let ξ tend to λ_0 and find $(\Delta - \lambda_0)^{-n} w_{\lambda} \in U$. Next let $(\Delta - \xi)^n v = \sum_{\lambda} w_{\lambda}^{\xi}$ and note that w_{λ}^{ξ} depends continuously on ξ . As

$$v = (\Delta - \xi)^{-n} (\Delta - \xi)^n v = \sum_{\lambda} (\Delta - \xi)^{-n} w_{\lambda}^{\xi},$$

we deduce $v_{\lambda} = (\Delta - \xi)^{-n} w_{\lambda}^{\xi}$ by uniqueness. For $\lambda \neq \lambda_0$ we let ξ tend to λ_0 and we can deduce $v_{\lambda} = (\Delta - \lambda_0)^{-n} w_{\lambda} \in U$. This implies $v - v_{\lambda} \in U$ as claimed. The rest of part (a) is clear.

For (b),(c) and (d) we argue that for an admissible representation the property (d) implies (b) and (c). To see that (d) implies (b) we consider a maximal π -filtration

$$F'_1 \subset F_1 \subset F'_2 \subset F_2 \subset \cdots \subset F'_l \subset F_l$$

and a complete G -stable L -filtration $(S_j)_{j \in L}$ with irreducible quotients. We claim that there must exist indices $\nu_1 < \nu'_1 < \cdots < \nu_l < \nu'_l$ in L such that $S_{\nu_i}/S_{\nu'_i} \cong \pi$ and $S_{\nu_i}/S_{\nu_{i-1}}$ has no π -sub-quotient, so that l equals the number of π -sub-quotients within the given L -filtration and this independent of the chosen maximal π -filtration. If the L -filtration is finite, this is the classical Jordan-Hölder Theorem. We reduce the present case to a finite filtration as follows: We choose a $\lambda \in \text{Spec}_{\pi}(\Delta)$. Then $(S_j^{\lambda} = S_j \cap V(\Delta, \lambda))_j$ is a filtration of this finite dimensional space. There must exist two neighboring indices $i_1 < j_1$ such that $S_{i_1}^{\lambda} = 0$ and $S_{j_1}^{\lambda} \neq 0$. Repeating we find indices $i_1 < j_1 < i_2 < j_2 < \cdots < i_k < j_k$ such that i_{ν} and j_{ν} are neighbored for each ν and $S_{j_{\nu}}^{\lambda} = S_{i_{\nu+1}}$ always holds, which implies that $S_{i_{n+1}}/S_{j_{\nu}}$ has no π -sub-quotient. Further S_{i_1} and V/S_{j_k} both have no π -sub-quotient. Now one can ignore the ν with $S_{j_{\nu}}/S_{i_{\nu}} \not\cong \pi$ and assume that all quotients are $\cong \pi$. From here on the classical proof of the Jordan-Hölder Theorem applies to show that $k = l$. After that, once we know that $N_{\Gamma, \omega}(\pi)$ equals the number of π -sub-quotients in the given L -filtration, part (c) also follows.

So it remains to show (d). Let $\lambda \in \text{Spec}_R(\Delta)$. By Zorn's lemma there exists a maximal G -stable subspace V_0 such that $V_0 \cap V(\Delta, \lambda) = 0$. Then its closure

\overline{V}_0 is admissible and thus satisfies the same claim, i.e., $\overline{V}_0 \cap V(\Delta, \lambda) = 0$, so by maximality, V_0 is closed. Let $v \in V(\Delta, \lambda)$ and let $S(v)$ denote the closure of the span of $V_0 + R(G)v$. Among all spaces $S(v)$ as v varies in $V(\Delta, \lambda) \setminus \{0\}$, there is a minimal one V_1 . Then V_1/V_0 is irreducible. We repeat the process with V/V_1 in the role of V , but with the same λ and repeat again until λ is no spectral value any more. We end up with a finite filtration

$$F_0 \subset \cdots \subset F_k$$

such that F_j/F_{j-1} is irreducible for $1 \leq j \leq k$ and λ is no spectral value of F_0 and V/F_k . We repeat the process with another spectral value and the spaces F_0 and V/F_k . One after the other, we thus eliminate the spectral values, which are countable in number, so in the limit we end up with a complete filtration with isolated limits as claimed. \square

2 The trace formula

Let G be a locally compact group and let $\Gamma \subset G$ be a cocompact lattice. This means that Γ is a discrete subgroup such that the quotient $\Gamma \backslash G$ is compact. Let $\omega : \Gamma \rightarrow \text{GL}(V)$ be a group homomorphism, where $V = V_\omega$ is a finite-dimensional complex vector space. Let $E = E_\omega = \Gamma \backslash (G \times V_\omega)$, where Γ acts diagonally. The projection onto the first factor makes E a vector bundle over $\Gamma \backslash G$. The space $\Gamma(E)$ of continuous sections can be identified with the space $C(\Gamma \backslash G, \omega)$ of all continuous functions $f : G \rightarrow V_\omega$ such that $f(\gamma x) = \omega(\gamma)f(x)$ for all $\gamma \in \Gamma$. Choose a hermitian metric on E to define the space $L^2(E)$ of L^2 -sections. This space can be identified with the space $L^2(\Gamma \backslash G, \omega)$ of all measurable functions $f : G \rightarrow V_\omega$ with $f(\gamma x) = \omega(\gamma)f(x)$ and $\int_F \langle f(x), f(x) \rangle_x dx < \infty$, where $F \subset G$ is a compact fundamental domain for $\Gamma \backslash G$. The group G acts by right translations on the Hilbert space $L^2(\Gamma \backslash G, \omega)$. This representation is continuous but in general not unitary. Let $R = R_{\Gamma, \omega}$ denote the right regular representation of G on the Hilbert space $H = L^2(\Gamma \backslash G, \omega)$.

Lemma 2.1. *Let G be a Lie group and $\Gamma \subset G$ a cocompact lattice. Fix a group-Laplacian Δ . Then the representation (R, H) with $H = L^2(\Gamma \backslash G, \omega)$ is Δ -admissible.*

Proof. The element $\Delta \in U(\mathfrak{g})$ acts on $C^\infty(\Gamma \backslash G, \omega)$ as a differential operator of order two whose principal symbol equals the square of the norm given by the Riemannian metric, such operators are called *generalized Laplacians* in [BGV04]. By [Shu01, Theorems 8.4 and 9.3] and [Mar88, Theorem 4.3] it follows that Δ has discrete spectrum in $L^2(\Gamma \backslash G, \omega)$, i.e., there exists a sequence λ_j of complex numbers which do not accumulate in \mathbb{C} such that the space $\bigoplus_{j=1}^\infty H(\Delta, \lambda_j)$ is dense in $H = L^2(\Gamma \backslash G, \omega)$. Each $v \in H$ can uniquely be written as convergent $\sum_j u_j$ with $u_j \in H(\Delta, \lambda_j)$.

One sets Λ_R equal to $\mathbb{C} \setminus \{\lambda_j : j \in \mathbb{N}\}$. Then for given $\lambda \in \Lambda_R$ the space $H(\Delta, \lambda)$ which lies in $C^\infty(\Gamma \backslash G, \omega)$, is finite-dimensional. The only tricky point is to show that for a given closed G -stable subspace $U \subset H$ one has $(\Delta - \xi)^{-1}U \subset U$. For this note that $(\Delta - \xi)^{-1} = f(\sqrt{\Delta})$ with $f(x) = (x^2 - \xi)^{-1}$. The Fourier transform of f is $\hat{f}(x) = \frac{e^{i|x|\sqrt{\xi}}}{2\sqrt{\xi}}$, where $\sqrt{\xi}$ denote the unique complex number α with $\text{Im}(\alpha) > 0$ and $\alpha^2 = \xi$. Let χ_0 be a smooth function on \mathbb{R} with $0 \leq \chi \leq 1$, $\chi(t) = 1$ for $t \leq 0$ and $\chi_1(t) = 0$ for $t \geq 1$. For $T > 0$ set $\chi_T(t) = \chi_0(t - T)$ and let f_T be defined by $\hat{f}_T(x) = \chi_T(|x|)\hat{f}(x)$. Then $f_T(x)$ has compact support and by [CGT82] it follows that the operator $f_T(\sqrt{\Delta})$ has finite propagation speed. We can view this operator of G or on $\Gamma \backslash G$. The connection between the two is as follows: On G this operator is invariant under left translation by elements of G , hence it is given by right convolution with a function, which, by finite propagation speed, has compact support. This function is continuous on G and smooth on the set $G \setminus \{1\}$. We denote it by $x \mapsto f(\sqrt{\Delta})(x)$. Then on $\Gamma \backslash G$ the operator $f(\sqrt{\Delta})$ has continuous kernel $k(x, y) = \sum_{\gamma \in \Gamma} f_T(x^{-1}\gamma y)$, the sum being locally finite. For $\phi \in L^2(E)$ one has $R(f(\sqrt{\Delta}))\phi(x) = \int_G f(\sqrt{\Delta})(y)\phi(xy) dy$ and approximating this integral by Riemann sums, one sees that $R(f_T(\sqrt{\Delta}))\phi$ lies in U if $\phi \in U$. It therefore suffices to show that $R(f_T(\sqrt{\Delta}))\phi$ converges to $R(\Delta - \xi)^{-1}\phi$ as $T \rightarrow \infty$. On the compact manifold $\Gamma \backslash G$ this follows if we show that the kernel of the former converges uniformly to the kernel of the latter, which is a consequence of Theorem 1.4 of [CGT82]. \square

Definition 2.2 (Test functions). Here we repeat the definition of the space of test functions as in [Bru61]. First, if L is a Lie group, then $C_c^\infty(L)$ is defined as the space of all infinitely differentiable functions of compact support on L . The space $C_c^\infty(L)$ is the inductive limit of all $C_K^\infty(L)$, where $K \subset L$ runs through all compact subsets of L and $C_K^\infty(L)$ is the space of all smooth functions supported inside K . The latter is a Fréchet space equipped with

the supremum norms over all derivatives. Then $C_c^\infty(L)$ is equipped with the inductive limit topology in the category of locally convex spaces as defined in [SW99], Chap II, Sec. 6.

Next, suppose the locally compact group H has the property that H/H^0 is compact, where H^0 is the connected component. Let \mathcal{N} be the family of all compact normal subgroups $N \subset H$ such that H/N is a Lie group. We call H/N a *Lie quotient* of H . Then, by [MZ55], the set \mathcal{N} is directed by inverse inclusion and

$$H \cong \varprojlim_N H/N,$$

where the inverse limit runs over the set \mathcal{N} . So H is a projective limit of Lie groups. The space $C_c^\infty(H)$ is then defined to be the sum of all spaces $C_c^\infty(H/N)$ as N varies in \mathcal{N} . Then $C_c^\infty(H)$ is the inductive limit over all $C_c^\infty(L)$ running over all Lie quotients L of H and so $C_c^\infty(H)$ again is equipped with the inductive limit topology in the category of locally convex spaces.

Finally to the general case. By [MZ55] one knows that every locally compact group G has an open subgroup H such that H/H^0 is compact, so H is a projective limit of connected Lie groups in a canonical way. A Lie quotient of H then is called a *local Lie quotient* of G . We have the notion $C_c^\infty(H)$ and for any $g \in G$ we define $C_c^\infty(gH)$ to be the set of functions f on the coset gH such that $x \mapsto f(gx)$ lies in $C_c^\infty(H)$. We then define $C_c^\infty(G)$ to be the sum of all $C_c^\infty(gH)$, where g varies in G . Then $C_c^\infty(G)$ is the inductive limit over all finite sums of the spaces $C_c^\infty(gH)$. Note that the definition is independent of the choice of H , since, given a second open group H' , the support of any given $f \in C_c(G)$ will only meet finitely many left cosets gH'' of the open subgroup $H'' = H \cap H'$. It follows in particular, that $C_c^\infty(G)$ is the inductive limit over a family of Fréchet spaces. This concludes the definition of the space $C_c^\infty(G)$ of test functions.

Theorem 2.3 (Trace formula). *Let G be a locally compact group and let $\Gamma \subset G$ be a cocompact lattice. Let (ω, V_ω) be a representation of the discrete group Γ on a finite-dimensional complex vector space V_ω and define the Hilbert space $H = L^2(\Gamma \backslash G, \omega)$ as above. Then for each $f \in C_c^\infty(G)$ the operator $R(f)$ is trace class and its trace equals either side of the equation*

$$\sum_{\pi \in \tilde{G}} N_{\Gamma, \omega}(\pi) \operatorname{tr} \pi(f) = \sum_{[\gamma]} \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \mathcal{O}_\gamma(f) \operatorname{tr} \omega(\gamma),$$

where $N_{\Gamma, \omega}(\pi)$ denotes the maximal length of a π -filtration in H , the sum on the right runs over all conjugacy classes $[\gamma]$ in Γ , the groups G_γ and Γ_γ are the centralizers of γ in G and Γ and \mathcal{O}_γ denotes the orbital integral

$$\mathcal{O}_\gamma(f) = \int_{G_\gamma \backslash G} f(x^{-1}\gamma x) dx.$$

The left hand side of the formula is also called the spectral side and the right hand side is the geometric side.

Proof. First assume that G is a Lie group. By the Theorem of Dixmier and Malliavin [DM78], every $f \in C_c^\infty(G)$ is a finite sum of convolution products $g * h$ with $g, h \in C_c^\infty(G)$. If $f = g * h$ then $R(f) = R(g)R(h)$. Now the same calculus as in the unitary case [DE09, Chapter 9] implies that $R(f)$ is an integral operator with smooth kernel $k(x, y) = \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y)\omega(\gamma)$, so by [DE09, Proposition 9.3.1] it is trace class and its trace equals $\int_{\Gamma \backslash G} \text{tr } k(x, x) dx$, which with the same computation as in the proof of [DE09, Theorem 9.3.2] is seen to be equal to

$$\sum_{[\gamma]} \text{vol}(\Gamma_\gamma \backslash G_\gamma) \mathcal{O}_\gamma(f) \text{tr } \omega(\gamma).$$

We thus get the geometric side of the trace formula. The spectral side is obtained from Theorem 1.8.

To finish the proof, we generalize the trace formula to arbitrary locally compact groups. So assume now that G is the projective limit of its Lie quotients,

$$G = \varprojlim_N G/N.$$

A given $f \in C_c^\infty(G)$ will factorize over some Lie quotient G/N . We can assume the compact group N chosen so small that $N \cap \Gamma = \{1\}$. Then Γ induces a cocompact lattice in G/N and the trace formula for this group implies the trace formula for the given f .

Finally, assume that trace formula holds for an open subgroup H of G , then $\Gamma \cap H$ is a cocompact lattice in H and the trace formula for H implies the trace formula for G . \square

3 Semisimple Lie groups

In the case of a semisimple group G we here prove a slightly stronger spectral theorem which says that the right regular representation on $L^2(E)$ is a direct sum of representations of finite length.

Definition 3.1. A representation (R, V) of a topological group has *finite length*, if there exists a filtration

$$0 = F_0 \subset \cdots \subset F_k = V$$

of closed G -stable subspaces such that F_j/F_{j-1} is irreducible for each j . The classical Jordan-Hölder Theorem says that then the irreducible quotients F_j/F_{j-1} are uniquely determined by V up to order.

Definition 3.2. We say that a representation (R, V) of a topological group G is a *Jordan-Hölder representation*, if it is a direct sum of finite length representations. More precisely, we insist that there are closed G -stable subspaces V_i , $i \in I$ such that the direct sum $\bigoplus_{i \in I} V_i$ is dense in V .

Let G be a semisimple Lie group with finite center and let K be a maximal compact subgroup. Let $\Gamma \subset G$ be a cocompact lattice and let (χ, V_χ) be a finite dimensional complex representation of Γ . Then χ defines a vector bundle $E = E_\chi$ over $\Gamma \backslash G$. The smooth sections can be described as

$$\Gamma^\infty(E) \cong (C^\infty(G) \otimes V_\chi)^\Gamma.$$

The choice of a hermitian metric on E allows the definition of the Hilbert space $L^2(E)$ of square integrable sections. We equip $\Gamma^\infty(E)$ with the topology of $L^2(E)$.

Let V_{fin} be the space of all sections in $\Gamma^\infty(E)$ which are K -finite as well as \mathfrak{z} -finite, where \mathfrak{z} is the center of the universal covering algebra $U(\mathfrak{g}_\mathbb{C})$ of the complexified Lie algebra $\mathfrak{g}_\mathbb{C}$ of G .

Theorem 3.3. *The (\mathfrak{g}, K) -module V_{fin} is dense in $\Gamma^\infty(E)$ as well as in $L^2(E)$. The G -representations on $\Gamma^\infty(E)$ and on $L^2(E)$ are Jordan-Hölder representations.*

Proof. For every $(\tau, V_\tau) \in \hat{K}$ the Casimir element $C \in \mathfrak{z}$ acts on the τ -isotype

$$\Gamma^\infty(E)(\tau) \cong V_\tau \otimes \text{Hom}_K(V_\tau, \Gamma^\infty(E)),$$

as it acts on

$$\begin{aligned} \text{Hom}_K(V_\tau, \Gamma^\infty(E)) &\cong (\Gamma^\infty(E) \otimes V_\tau)^K \\ &\cong (C^\infty(G) \otimes V_\chi \otimes V_\tau)^{\Gamma \times K} \\ &\cong \Gamma^\infty(E_{\chi, \tau}), \end{aligned}$$

where $E_{\chi, \tau}$ is the vector bundle over $\Gamma \backslash G / K$ defined by $\chi \times \tau$. On $\Gamma^\infty(E_{\chi, \tau})$ the Casimir C induces an operator which has the same principal symbol as the Laplacian for any given metric. Hence ([Shu01], Theorems 8.4 and 9.3) the operator C has discrete spectrum on $L^2(E_{\chi, \tau})$ consisting of eigenvalues of finite multiplicity.

Let $\lambda \in \mathbb{C}$ be an eigenvalue and let $\Gamma^\infty(E_{\chi, \tau})(\lambda)$ be the corresponding finite dimensional generalized eigenspace. The image $V_{\tau, \lambda}$ of $\Gamma^\infty(E_{\chi, \tau})(\lambda)$ in $\Gamma^\infty(E)$ is \mathfrak{z} -stable and K -stable. Hence the generated (\mathfrak{g}, K) -module $U(\mathfrak{g})V_{\tau, \lambda}$ is in V_{fin} and by Corollary 3.4.7 of [Wal88] it is admissible and as it is finitely generated, it is a Harish-Chandra module, so by Corollary 10.42 of [Kna01] it has a finite composition series:

$$U(\mathfrak{g})V_{\tau, \lambda} = F_k \supset F_{k-1} \supset \cdots \supset F_0 = 0$$

with irreducible quotients F_{j+1}/F_j . We repeat this argument with a different K -type τ' not occurring in $U(\mathfrak{g})V_{\tau, \lambda}$ if it exists. Otherwise, we repeat it with a different eigenvalue λ to get the claim. \square

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